STUDY OF THE PROBABILITY OF SNAP-THROUGH OF A LONG, CYLINDRICAL PANEL UNDER RANDOM PRESSURE

(ISSLEDOVANIE VEROIATNOSTI KHLOPKA UDLINENNOI TSILINDRICHESKOI PANELI POD DEISTVIEM SLUCHAINOGO DAVLENIIA)

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By means of the statistical theory of shell stability [1,2], the specific problem is considered of the probability of snap-through, if the load lies between the lower and upper critical values. The shell parameters are taken as fixed, and the probability of snap-through is postulated as due to the presence of exciting forces which are representative of a certain stationary random process. The solution is connected with the study of the nonlinear forced oscillations of a shell (or other elastic system having the possibility of jumps) under random forces, and from this point of view has an added interest for such problems as may arise, for example, in the theory of aeroelasticity. It is presupposed that during variations in the generalized coordinates and velocities the probable after-effects are missing. It is known that this is true if the spectral density of the excitation is constant in the region of the natural frequencies (white noise excitation) [3]. One may treat the shell problem as a system with one degree of freedom and so the idea of the absence of probable after-effects is admissible to a first approximation during the evolution of the coordinates and velocities if the excitation has the properties of white noise. The excitation for this spectral density must be described by the value corresponding to the natural frequencies. It is necessary to know the error of such a method in the investigation. As a basis one assumes that it decays strongly with decrease in damping in the system and with the correlation time of the excitation [4].

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1. We consider a long hollow cylindrical panel of radius R, width b, thickness h, compression nodulus and Poisson's ratio E and ν. It will be assumed that the edges $y = 0$ and $y = b$ (coordinate y along the arc) are hinged to immovable ribs. The shell is under uniformly distributed pressure of intensity q which is a random function of time. The problem consists in the determination of the probability of snap-through of the shell during an interval of time Δt.

Consider a panel with not too large a value of the curvature parameter

$$k = b^2 / Rh \tag{1.1}$$

Assume [5] the deflection to be approximated with sufficient accuracy by the expression

$$w = h\zeta \sin(\pi y / b) \tag{1.2}$$

After calculation by the usual method, we obtain for the potential energy of the shell and the external forces ϑ per unit length of shell, and for the kinetic energy T

$$\vartheta = \frac{Eh^5}{b^3(1-\nu^2)}\vartheta^*, \quad \vartheta^* = \frac{\pi^4}{32}\zeta^4 - \frac{\pi}{2}k\zeta^3 + \left(\frac{\pi^4}{48} + \frac{2k^2}{\pi^2}\right)\zeta^2 - \frac{2}{\pi}\lambda\zeta, \quad T = b\rho\frac{h^2}{4}\dot{\zeta}^2 \tag{1.3}$$

Here ρ is the mass per unit area of middle surface, and

$$\lambda = \frac{q}{E}\left(\frac{b}{h}\right)^4(1-\nu^2) \tag{1.4}$$

The stochastic differential equation of motion including the dissipation term is, in Lagrangian form

$$\ddot{\zeta} = -\beta\rho^{-1}\dot{\zeta} - \frac{2Eh^3}{b^4(1-\nu^2)\rho}\frac{d\vartheta^*}{d\zeta} \tag{1.5}$$

The coefficient $\beta\rho^{-1}$ may be expressed in terms of the decrement Δ and the frequency ω of natural oscillations about the undeformed state

$$\beta\rho^{-1} = \omega\Delta\pi^{-1} \tag{1.6}$$

It is easy to see that

$$\omega^2 = \frac{Eh^3}{b^4(1-\nu^2)\rho}\left(\frac{\pi^4}{12} + \frac{8k^2}{\pi^2}\right) \tag{1.7}$$

Further, we set

$$q = Mq + \xi(t) \qquad (Mq = \text{const}) \tag{1.8}$$

and treat Mq as a determinate part of the load; the fluctuation $\xi(t)$ represents a centered stationary random process with a known spectral density $f_\xi(\omega) = f$ as its excitation. Equation (1.5) may be written, taking account of (1.8), as

$$\ddot{\zeta} = \beta\rho^{-1}\dot{\zeta} - \frac{dV}{d\zeta} + \frac{4}{\pi h\rho}\xi, \qquad V = \frac{2Eh^3}{b^4(1-\nu^2)\rho}\vartheta^*(M\lambda) \tag{1.9}$$

We introduce a density of probability distribution $p(t, \zeta, \dot{\zeta})$ determined in the phase plane of the variables $\zeta, \dot{\zeta}$. By virtue of the arguments previously proposed, it is assumed that the process $(\zeta, \dot{\zeta})$ is one without after-effects (i.e. a Markov process), and that the function $p(t, \zeta, \dot{\zeta})$ must satisfy the known equation of Kolmogorov [6] which in this case has the form

$$\frac{\partial p}{\partial t} = -\dot{\zeta}\frac{\partial p}{\partial \zeta} - \frac{\partial}{\partial \dot{\zeta}}\left[\left(-\beta\rho^{-1}\dot{\zeta} - \frac{dV}{d\zeta}\right)p\right] + a\frac{\partial^2 p}{\partial \dot{\zeta}^2}, \qquad a = \frac{16f}{\pi\rho^2 h^2} \tag{1.10}$$

As has been shown, the function f must be chosen in accordance with the spectral density of the exciting pressure $\xi(t)$ which corresponds to the frequency ω. The equation of Vorovich [2] is obtained from Equation (1.10) if the effects of the order of duration $\rho\beta^{-1}$ are neglected and if variations of the coordinates are considered as a Markov process.

It is easy to convenice oneself that Equation (1.10) has a stationary solution

$$p_0 = C\exp\left[-\frac{\beta}{\rho a}\left(\frac{\dot{\zeta}^2}{2} + V\right)\right] \tag{1.11}$$

An analogy with the Maxwell-Boltzmann distribution in the presence of a force field may be noted. The corresponding stationary solution of the Vorovich equation has, at the same time, the form of a Gibbs distribution.

2. For $\lambda_- < M\lambda < \lambda_+$, where λ_- and λ_+ are the lower and upper critical values, the shell has three equilibrium states $\zeta_1, \zeta_2, \zeta_3$ $(\zeta_1 < \zeta_2 < \zeta_3)$. They are singular points of Equation (1.9) for $\dot{\zeta} = 0$ and roots of the equation of equilibrium

$$dV / d\zeta = 0 \tag{2.1}$$

The equilibrium at points ζ_1 and ζ_3 is stable according to Liapunov, and unstable at ζ_2. We suppose that initially at time t_0 the shell is in an unexcited state ζ_1 with no initial velocity. Our problem consists in determining the probability of snap-through during the interval $[t_0, t_0 + \Delta t]$. First of all we must distinguish the probability of the shell overshooting the excited equilibrium state ζ_3, surmounting a potential barrier $H = V(\zeta_2) - V(\zeta_1)$, from the probability that the shell will be found in the neighborhood of the state ζ_3 at time $t_0 + \Delta t$.

The first of these probabilities (a function of Δt and the initial values of the coordinates and velocities) satisfies the equation conjugate to (1.10) and the corresponding boundary conditions. It reduces to unity as $\Delta t \to \infty$. There remains the second probability, denoted by $P(*/\Delta t)$.

It is known [7] that any distribution satisfying Equation (1.10) reduces to the distribution (1.11) as $t \to \infty$. Therefore, for $\Delta t > \tau$, where τ is the relaxation time, the probability $P(*/\Delta t)$ must be calculated according to the distribution (1.11). Vorovich [2] has shown a method for such a calculation.

Of course the equilibrium distribution (1.11) and so the probability $P(*/\Delta t)$ calculated in this way give no information on the history of the shell during the relaxation time. Therefore we determine the probability of the first snap-through taking Δt small by comparison with τ. In this case both of the above mentioned probabilities coincide. We make use of the following considerations laid down by Kramers [8] for finding $P(*/\Delta t)$. We shall consider that the function $p(t, \zeta, \dot{\zeta})$ represents an ensemble in the sense of classical statistical physics (solely for convenience in terminology).

In correspondence with initial conditions we may set

$$p(t_0, \zeta, \dot{\zeta}) = \delta(\zeta - \zeta_1)\,\delta(\dot{\zeta}) \tag{2.2}$$

We describe the process of transformation of the distribution (2.2) into the distribution (1.11). At the point ζ_1 there is a profusion of ensemble systems for $t \approx t_0$ by comparison with (1.11), (with an insufficiency at the point ζ_3), and so in the transformation process there will occur diffusion of the ensemble systems from ζ_1 to ζ_3 through ζ_2. The probability $P(*/\Delta t)$ may be determined as follows by

$$P(*/\Delta t) = \frac{j\Delta t}{n} \tag{2.3}$$

where j is the diffusion flow per unit time and n is the number of ensemble systems in the neighborhood of the point ζ_1.

For determination of the probability of the first snap-through it is natural to take the ensemble density at the neighborhood of ζ_3 negligibly small by comparison with that in the neighborhood of ζ_1. In addition, we limit ourselves to force excitations of comparatively small energy, satisfying the condition

$$\rho a \ll \beta H \tag{2.4}$$

This assumption gives the possibility of calculating the diffusion flows as stationary and slow in the sense that the appearance of snap-through precedes the establishment of the equilibrium distribution (1.11) in the neighborhood of ζ_1. Consequently, the relation

$$j = \int_{-\infty}^{\infty} p_1(\zeta_2, \dot{\zeta})\, \dot{\zeta} d\dot{\zeta} \tag{2.5}$$

is justified. Here $p_1(\zeta, \dot{\zeta})$ satisfies Equation (1.10) near ζ_2 and the conditions

$$p_1 \to p_0 \quad \text{for } \zeta \ll \zeta_2, \qquad p_1 \to 0 \quad \text{for } \zeta \gg \zeta_2 \tag{2.6}$$

It is easy to convince oneself that the function

$$p_1 = C \exp\left[-\frac{\beta}{\rho a} V(\zeta_2)\right] \sqrt{\frac{\alpha - \beta\rho^{-1}}{2\pi a}} \exp\left[-\frac{\beta}{2\rho a}\left(\dot{\zeta}^2 + \left(\frac{d^2V}{d\zeta^2}\right)_{\zeta_2}(\zeta - \zeta_2)^2\right)\right] \times$$
$$\times \int_{-\infty}^{\dot{\zeta} - a(\zeta - \zeta_2)} \exp\left[-(\alpha - \beta\rho^{-1})\frac{x^2}{2a}\right] dx, \qquad \alpha = \frac{\beta}{2\rho} + \sqrt{\frac{\beta^2}{4\rho^2} - \left(\frac{d^2V}{d\zeta^2}\right)_{\zeta_2}} \tag{2.7}$$

fulfills these requirements.

Upon substitution of (2.7) into (2.7) and integrating, we find that

$$j = C\left(\frac{a\rho}{\beta}\right) \sqrt{\frac{\alpha - \beta\rho^{-1}}{\alpha}} \exp\left[-\frac{\beta}{a\rho} V(\zeta_2)\right] \tag{2.8}$$

For the calculation of n, it is supposed that the equilibrium distribution (1.11) is realized near ζ_1, as shown. We have

$$n \approx C \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left\{-\frac{\beta}{\rho a}\left[\frac{\dot{\zeta}^2}{2} + V(\zeta_1) + \frac{1}{2}\left(\frac{d^2V}{d\zeta^2}\right)_{\zeta_1}(\zeta - \zeta_1)^2\right]\right\} d\zeta\, d\dot{\zeta} =$$
$$= C \exp\left[-\frac{\beta}{\rho a} V(\zeta_1)\right] 2\pi \frac{a\rho}{\beta} \sqrt{\left(\frac{d^2V}{d\zeta^2}\right)_{\zeta_1}} \tag{2.9}$$

By substitution of (2.8) and (2.9) into (2.3) we obtain

$$P(*/\Delta t) = \frac{\Delta t}{2\pi} \sqrt{-\frac{(d^2V/d\zeta^2)_{\zeta_1}}{(d^2V/d\zeta^2)_{\zeta_2}}}\left[\sqrt{\frac{\beta^2}{4\rho^2} - \left(\frac{d^2V}{d\zeta^2}\right)_{\zeta_2}} - \frac{\beta}{2\rho}\right] \exp\left(-\frac{\beta}{\rho a} H\right) \tag{2.10}$$

Formula (2.10) is obtained from a representation of the $(\zeta, \dot{\zeta})$ process as a Markov process. If, following Vorovich [2], effects are neglected which occur during an interval of the order of $\rho\beta^{-1}$ and if ζ is taken as a Markov process, then analogous considerations give [9]

$$P(*/\Delta t)=\frac{\Delta t\rho}{2\pi\beta}\sqrt{-\left(\frac{d^2V}{d\zeta^2}\right)_{\zeta_1}\left(\frac{d^2V}{d\zeta^2}\right)_{\zeta_2}}\exp\left(-\frac{\beta}{\rho a}H\right) \qquad (2.11)$$

It is known, for example, from (1.6), that $\Delta = 0.005 - 0.05$ for steel construction. It follows that in this case by using (2.11) we neglect effects occurring in intervals of the order of several tens of periods of the natural oscillations.

We reemphasize that Formulas (2.10) and (2.11) are valid for $\Delta t \ll \tau$. The relaxation time may be determined as the time required for the ensemble density at ζ_3 to achieve the value calculated from (1.11) as a result of the diffusion flow j. It is evident that the larger j, the smaller τ. It may be concluded from Formula (2.8) that the relaxation time decreases with increasing a, i.e. the spectral density of the excitation, and increases with increasing Δ, the damping. According to Formula (2.8) the quantitative estimation of τ is not possible since the formula holds only at instants of time when the ensemble density is still negligibly small at ζ_3.

We consider, further, the probability $P(*/2\pi\omega^{-1})$ relative to the period $(2\pi\omega^{-1})$ of the natural oscillations. We introduce the dimensionless parameters

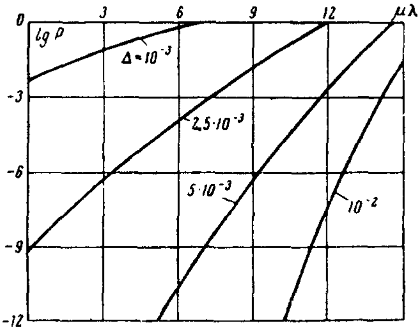
$$\delta = h/b, \qquad \varphi = f\omega / E^2 \qquad (2.12)$$

Formulas (2.10) and (2.11) take the form

$$P(*/2\pi\omega^{-1})=\sqrt{-\frac{(d^2\vartheta^*/d\zeta^2)_{\zeta_1}}{(d^2\vartheta^*/d\zeta^2)_{\zeta_2}}}\left[\sqrt{\frac{\Delta^2}{4\pi^2}-\frac{2}{\Lambda}\left(\frac{d^2\vartheta^*}{d\zeta^2}\right)_{\zeta_2}}-\frac{\Delta}{2\pi}\right]\exp\frac{-\delta^8\Delta\Lambda H^*}{2(1-\nu^2)^2\varphi} \qquad (2.13)$$

$$P(*/2\pi\omega^{-1})=\frac{2\pi}{\Delta\Lambda}\sqrt{-\left(\frac{d^2\vartheta^*}{d\zeta^2}\right)_{\zeta_1}\left(\frac{d^2\vartheta^*}{d\zeta^2}\right)_{\zeta_2}}\exp\frac{-\delta^8\Delta\Lambda H^*}{2(1-\nu^2)^2\varphi} \qquad \left(\Lambda=\frac{\pi^4}{12}+\frac{8k^2}{\pi^2}\right) \qquad (2.14)$$

3. Below are given calculated results from Formulas (2.13) and (2.14) for an aluminum panel with $k = 10$, $h = 1.6$ mm, $\delta = 0.5 \times 10^{-2}$, and $\Delta t = 10 \times 2\pi\omega^{-1}$. The spectral density f was taken as the acoustical pressure on the fuselage of a "Comet" 1 airplane. Values of ζ_1 and ζ_2 were found graphically. The upper critical load λ_+ was equal to 19.1 and the lower critical load λ_- was negative.



The figure shows the relation between $P(*/\Delta t)$ calculated from (2.14) and $\mu\lambda$ for different values of Δ. It may be noted that there is a substantial probability of snap-through for loads less than λ_+. For example, for $\Delta = 10^{-3}$ the probability of snap-through is practically unity beginning with $M\lambda = 6.9$. We remark that Equation (2.4) is satisfied with sufficient accuracy for values of $M\lambda$ corresponding to points on the curves.

It has been noted above that the ratio s of the probability of snap-through from Formula (2.14) to the probability from the more exact Formula (2.13) must reduce to unity with increasing Δ. We present values of s calculated for certain values of Δ for three values of $M\lambda$:

$\Delta\, 10^3 =$ 2	4	6	8	20	40	60	
$s\, 10^{-2} =$ 20	9.8	6.6	4.8	2.0	0.98	0.65	$(M\lambda = 6.7)$
$s\, 10^{-2} =$ 19	9.5	6.3	4.7	1.9	0.95	0.63	$(M\lambda = 9.7)$
$s\, 10^{-2} =$ 18	9.0	6.0	4.5	1.8	0.89	0.61	$(M\lambda = 13)$

It is seen that within physically realizable limits of Δ Formula (2.14) raises the result appreciably.

It is easily concluded from the figure and from (2.13) and (2.14 that the probability of snap-through appears to be very sensitive to changes in the load $M\lambda$, the decrement Δ, the geometric and physical characteristics of the shell; and this may be the reason for the scatter of experimental data repeatedly observed.

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